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# Topological Conformal Field Theories and the Flat Coordinates

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ABSTRACT: We discuss the relation between the topological conformal field theories (TCFT) and the singularity theory. We argue that the problem of the explicit solution of these theories is equivalent to the problem of the finding of the flat coordinate basis on the space of couplings. The latter problem is the well known problem of the singularity theory. It has especially simple solution in the case of the minimal models.

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## 1. INTRODUCTION

The recent progress in the study of the low dimensional field theories and the matrix models [1] showed a number of remarkable connections between the two-dimensional quantum gravity, low dimensional string theories, hierarchies of integrable differential equations, matrix models, Liouville model and loop equations. In particular it was found that the study of the two dimensional quantum gravity is closely connected with the investigation of the topological field theories. The topological field theories were first introduced by Witten [2]. The topological field theories that play the crucial role in the study of the low dimensional string theories and matrix models are the so called topological conformal field theories introduced in refs. [3,4,5] . These models can be coupled to the topological gravity [6,7]. The connection between the topological field theories and the matrix models was first noticed in refs. [8,9] and investigated in detailes in the subsequent papers [10,11,12,13].

It was found recently that these topological field theories are closely related to the singularity theory [11,12,14] . The origin of this connection is the relation between the N=2 superconformal field theories and the singularity theory ( see e.g. refs. [15,16] ). This relation is based on the identification of the N=2 superconformal models as the fixed points of the Landau-Ginsburg field theories.

In particular the approach based on the singularity theory was applied in refs. [11,12] to the study of the minimal topological models at genus zero. It was shown that the physical amplitudes in the minimal topological conformal theories coupled to the two dimensional topological gravity [6,7] are the same as in the matrix models. Using the recursion relations [8,9,4] all physical amplitudes in the minimal topological field theory coupled to gravity can be expressed through the ampitudes in the genus zero minimal topological field thery. Moreover, it was noted in ref. [12] the that in order to find the physical amplitudes in the topological field theory coupled to gravity it is enough to find the three and two point ampitudes in the topological field theory before its coupling to gravity. The reason is that these couplings are not altered by the inclusion of gravity. Consequently, all physical amplitudes in the two dimensional topological field theory coupled to two dimensional topological gravity can be determined if we calculate the physical amplitudes in the topological field theory before its coupling to gravity.

All these results show that the study of the topological conformal field theories plays the key role in the understanding of the two dimensional gravity coupled to matter.

Up to now the investigation of the topological conformal theories concentrated mostly on the case of the minimal models. Some results concerning topological conformal theories including the models with the central charge $c > 3$ were obtained in ref. [14] where the connection between the topological field theories and the singularity theory was studied.

The purpose of the present paper is to clarify further the connection between the topological field theories and the singularity theory. We will show that the problem of the determination of the physical amplitudes (i.e. the correlation functions of the scaling operators) in the general topological conformal field theories is equivalent to the well studied problem in the singularity theory. This problem is the problem of finding the so called flat coordinates. In turn, the latter problem is closely connected with the study of the Gauss-Manin systems. These systems are the systems of the differential equations for the integrals of the basic differential forms over the vanishing cycles associated with a given singularity [17,18,19,20,21,22,23,24,25,26,27].

We show that the approach based on the study of the Gauss-Manin differential systems permits the simple determination of all physical amplitudes in the minimal topological models at genus zero. In particular the problem of finding the physical amplitudes reduces to the problem of transforming the system of Gauss-Manin differential equations into the simple form. We include the case of the $E_7$ and $E_8$ models that were not considered before in the study of the topological conformal field theories. Note that although the physical ampitudes for the $E_7$ and $E_8$ cases were not considered yet, the corresponding mathematical problem of finding the flat coordinates was solved by mathematicians long ago [24,28].

The singularity theory gives the unified picture for the structure of the generic topological conformal field theories, including those with the central charge bigger than one. Consequently, it gives the way to study the topological matter coupled to topological gravity. It seems that in the case of the general topological conformal field theory the Gauss-Manin differential equations play the role analogous to the role of the KDV hierarchy in the case of the minimal models.

The paper is organised in the following way. In section 2 we briefly review the basic definitions of the topological conformal field theory. We also review the coupling of the two dimensional topological field theories to the two dimensional topological gravity. In section 3 we discuss some basic definitions of the singularity theory and their connection with the topological conformal field theories. We review the notion of the flat coordinates and discuss how they can be used to calculate physical amplitudes in the general topological conformal field theories and the partition function in the topological field theory coupled to the topological gravity. Knowledge of the latter function permits one to calculate all physical ampitudes in the topological field theory coupled to gravity at least for the genus zero case. In section 4 we write the answers for the case of the minimal models including $E_7$ and $E_8$ theories. Section 5 is the summary.

## 2. TOPOLOGICAL CONFORMAL FIELD THEORIES

In this section we briefly review the basic formalism of the topological field theories [2,3,4,5,8,9,10,11,12,13,6,7,14].

We first recall the definition of the two-dimensional topological theories [2]. The main feature of the topological conformal theories is that they posess the nilpotent Q-symmetry:

$$Q^2 = 0. \tag{1}$$

The operator Q acts on the space of all states of the topological field theory.

The physical states in the topological field theory are characterised by the cohomology of the operator Q. The space $H$ of the physical states is equal to

$$H = \frac{KerQ}{ImQ}. \tag{2}$$

In other words, the local observables are defined by the relation:

$$Q|\phi_i >= 0. \tag{3}$$

Also

$$|\phi_i >\equiv |\phi_i > +Q|\lambda > . \tag{4}$$

The latter equation means that the correlators of the physical operators are independent of the representative of $\phi_i$.

The crucial property of the topological field theories is that the energy-momentum tensor is the commutator of Q with some other operator $G_{\alpha\beta}$

$$T_{\alpha\beta} = \{Q, G_{\alpha\beta}\}. \tag{5}$$

Consequently, the correlation functions of the physical operators are independent on the coordinates of the operator insertions.

The next important property of the topological field theories is the factorisation property. The correlation functions can be factorised by inserting the complete set of states in the intermediate channels. This property amounts to the equation:

$$1_{phys} = \sum_{i,j} |\phi_i > \eta^{ij} < \phi_j|, \tag{6}$$

where $\eta^{ij}$ is the metric on $H$. This metric is the inverse of the metric $\eta_{ij}$ defined as

$$< \phi_i \phi_j >= \eta_{ij}. \tag{7}$$

The class of the two dimensional topological field theories that play the special role in the study of the two dimensional gravity are the perturbed topological conformal field theories. First, let us define the topological conformal field theories. These are the topological field theories that are also conformally invariant, i.e. the energy-momentum tensor is traceless:

$$T^{\alpha}_{\alpha} = 0. \tag{8}$$

The combined presence of conformal invariance and the topological symmetry implies that the generator Q can be decomposed into holomorhic ( left) and the antiholomorphic (right) components that are dependent only on z and $\bar{z}$ . The way to obtain the topological conformal field theory is to start with the N=2 superconformal model with the central charge c. The topological conformal field theories are obtained from the N=2 superconformal theories by twisting the energy-momentum tensor in the way that the twisted energy-momentum tensor has the central charge equal to zero.

$$T'(z) = T(z) + 1/2\partial_z J(z). \tag{9}$$

Here $T'(z)$ is the energy-momentum tensor of the topological conformal field theory, $T(z)$ is the energy-momentum tensor of the original N=2 superconformal model and $J(z)$ is the U(1) current.

In addition to the nilpotent Q-symmetry the topological conformal field theories contain also the following holomorphic fields: the fermionic spin-2 field G and the U(1)-current J. These fields are the Q-partners of the energy-momentum tensor T. These fields are related by the following equations:

$$\begin{aligned} T &= \{Q, G\}, \\ Q &= -[Q, J] \end{aligned} \tag{10}$$

and

$$
\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n},\ [J_m, J_n] = dm\delta_{m+n,0},\\
&[L_m, G_n] = (m-n)G_{m+n},\ [J_m, G_n] = -G_{m+n},\\
&[L_m, Q_n] = -nQ_{m+n},\ [J_m, Q_n] = Q_{m+n},\\
&\{G_m, Q_n\} = L_{m+n} + nJ_{m+n} + \frac{1}{2}dm(m+1)\delta_{n+m,0},\\
&[L_m, J_n] = -nJ_{m+n} - \frac{1}{2}dm(m+1)\delta_{m+n,0}
\end{aligned}
\qquad . \tag{11}
$$

The parameter d is equal to $c/3$ and is identified with the target-space dimension. The currents G and Q are the modifications of the supercurrents $G^+$ and $G^-$ of the N=2 algebra of the original superconformal model. They have the Loran expansions of the form:

$$
\begin{aligned}
T(z) &= \sum_{-\infty}^{\infty} L_n z^{-n-2};\ G(z) = \sum_{-\infty}^{\infty} G_n z^{-n-2},\\
J(z) &= \sum_{-\infty}^{\infty} J_n z^{-n-1};\ Q(z) = \sum_{-\infty}^{\infty} Q_n z^{-n-1},
\end{aligned}
\tag{12}
$$

The primary operators in the topological conformal field theory correspond to the cohomology of Q defined by eqs. (3), (4) . These states are in one to one correspondence with the states in the chiral ring of the original N=2 superconformal model.

In order to specify uniquely the primary fields we must specify the additional constraints, e.g.:

$$G_0|\phi_i>= L_0|\phi_i>= 0. \tag{13}$$

These are precisely the conditions that define the chiral primary fields in the N=2 superconformal theories. These primary fields have the zero dimension relative to the energy momentum tensor (9).

Each primary field has the U(1) charge:

$$J_0|\phi_i>= q_i|\phi_i>, \quad 0 \le q_i \le d. \tag{14}$$

The U(1) charge is the conserved quantum number.

Consider now the family of the topological field theories defined by the parameter family of actions

$$S(t) = S(0) + \sum_{n=1}^{n=\mu} t_n \int \phi_n \tag{15}$$

Here we introduced the coupling constants $t_n$ for each scaling operator. Note that the operators $\phi_i$ do not depend on the variables $t_i$. The family of TCFT (15) corresponds to the versal deformation in the singularity theory and $t_i$ are the parameters of this deformation. These topological field theories are called the perturbed topological conformal field theories [12]. For shortness we shall usually omit the word "perturbed" in the future. The point $(t_1, ..., t_\mu) = (0, ..., 0)$ on the space of couplings corresponds to the unperturbed theory.

The main objects of interest in the topological conformal field theory are the functions $F$, $\eta_{ij}$, $c_{ijk}$. The function $F$ is defined as

$$F(t_1, ..., t_n) =< exp(\sum t_k \int \phi_k) > . \tag{16}$$

The function $\eta_{ij}$ is the metric on the space of the couplings $t_i$:

$$\eta_{ij}(t) =< \phi_i \phi_j exp(\sum t_k \int \phi_k) > . \tag{17}$$

The functions $c_{ijk}$ are the three-point correlation functions:

$$c_{ijk}(t) =< \phi_i \phi_j \phi_k exp(\sum t_n \int \phi_n) > . \tag{18}$$

Here the mean values denoted by the brackets $<>$ are calculated using the unperturbed action $S(0)$. All physical amplitudes of the theory can be expresed through the three-point correlation functions and the metric $\eta_{ij}$.

Three objects F, $\eta_{ij}$ and $c_{ijk}$ are connected by the differentiation over the coupling constants:

$$\begin{aligned} \eta_{ij}(t) &= \partial_i \partial_j \partial_1 F, \\ c_{ijk} &= \partial_i \partial_j \partial_k F. \end{aligned} \tag{19}$$

The many point correlation functions can be obtained either by the many time differentiation of F or can be expresed through $\eta_{ij}$ and $c_{ijk}$ using the factorisation properties of the correlation functions in the topological field theories (6) as e.g.:

$$\begin{aligned} <\phi_i\phi_j\phi_k\phi_l> &= \sum_m c_{ij}^m c_{mkl}, \\ <\phi_i\phi_j\phi_k\phi_l \int \phi_n> &= \sum_m (\partial_n c_{ij}^m c_{mkl} + c_{ij}^m \partial_n c_{mkl}). \end{aligned} \tag{20}$$

These results hold for the arbitrary topological conformal field theories. The indices in $c^i_{jk}$ are raised and lowered using the metric $\eta_{ij}$. Also, it follows from the U(1) charge conservation law that this metric is antidiagonal, that is $\eta_{ij} = 0$ if $q_i + q_j \neq d$.

Let us consider the correlation function

$$<\phi_{i_1}...\phi_{i_s} \int d^2z\phi_{i_{s+1}}... \int d^2z\phi_{i_{s+r}}> . \tag{21}$$

Here the integration goes over the whole complex plane. Only correlation functions that satisfy the U(1)-charge conservation law are nonzero:

$$\sum_{j=1}^{j=s} q_j + \sum_{j=s+1}^{j=r+s} (q_j - 1) = d. \tag{22}$$

It is possible to prove using the Ward identities associated with the current G that the metric $\eta_{ij}$ is constant [11,12]

$$\partial_k \eta_{ij} = 0. \tag{23}$$

The scaling properties of the perturbed correlators lead to the following equation for F:

$$\sum_j (q_j - 1) t_j \partial_j F(t) = (d-3) F(t) \tag{23}$$

that is $F(t_1, ..., t_\mu)$ is a quasihomogenius polynomial of the degree d-3 with the weights $q_j - 1$.

The coordinates $t_i$ form a distinguished basis in the space of couplings. They correspond to the directions in the space of all TCFT that are the pertubations by the scaling operators. However, we can take the arbitrary directions $s_1, ..., s_\mu$, where $s_j = s_j(t_1, ..., t_\mu)$; $s_j(0) = 0$, $j = 1, .., \mu$, on the coupling space. In this case we must change in the eqs. (19) the operation of the differentiation by the operation of the covariant differentiation defined by the metric $\eta_{ij}$. The three objects F, $\eta_{ij}$ and $c_{ijk}$ in the arbitrary smooth coordinates on the coupling space are connected by the operation of the covariant differentiation over the coupling constants:

$$\begin{aligned} \eta_{ij} &= \nabla_i \nabla_j \nabla_1 F, \\ c_{ijk} &= \nabla_i \nabla_j \nabla_k F. \end{aligned} \tag{24}$$

Here $\nabla_j$ is the covariant derivative operator respective to the metric $\eta_{ij}$:

$$\nabla_i A_j = \partial_i A_j + \Gamma^k_{ij} A_k, \tag{25}$$

where

$$\Gamma^i_{jk} = \eta^{ip}(\frac{\partial \eta_{pk}}{\partial s_j} + \frac{\partial \eta_{pj}}{\partial s_k} - \frac{\partial \eta_{jk}}{\partial s_p}) \tag{26}$$

So far we discussed only the perturbed topological conformal field theories that were not coupled to the topological gravity. We shall not discuss the coupling to gravity here in detailes (see e.g. ref. [12]). We only note that the complete set of the BRST invariant operators in the coupled system is given by

$$\sigma_{n,\alpha} = \phi_\alpha \gamma_0^n P. \tag{27}$$

Here P is the puncture operator, $\phi_\alpha$ is the chiral primary field from the matter sector and $\gamma_0$ is the basic BRST invariant operator from the ghost sector.

The important property of the correlation functions in the toplogical theory coupled to gravity is the independence of the three point functions from the coupling to gravity. Namely,

$$\begin{aligned} &< \sigma_{0,\alpha} \sigma_{0,\beta} \sigma_{0,\gamma} >=< \phi_\alpha \phi_\beta \phi_\gamma >, \\ &< P \sigma_{0,\alpha} \sigma_{0,\beta} >=< \phi_\alpha \phi_\beta > . \end{aligned} \tag{28}$$

The l.h.s. of eqs. (28) is calculated in the presence of the two-dimensional topological gravity, while the r.h.s. is calculated in the topological theory before switching the gravity.

The function F in eq. (16) is the partition function of the topological field theory coupled to gravity [12]. All physical amplitudes in the topological theory coupled to gravity, at least for genus zero, can be determined if we know the partition function F. The amplitudes with $\sigma_{n,\alpha}$, $n > 0$, can be expressed through the amplitudes with $\sigma_{0\alpha}$ that figure in eq. (28) using the recursion relations [12]. These recursion relations do not depend on the central charge d of the topological field theory and we shall not write them here explicitly.

We see that in order to solve (i.e. to determine all physical amplitudes) the topological field theory coupled to gravity at genus zero it is enough to determine the three and two point correlation functions in the toplogical theory before coupling it to gravity. Integrating these correlation functions three times we get F (see eq. (19)).

We shall now discuss the relation between these correlation functions and the singularity theory [14]. Denote by w$(x_1, ..., x_n)$ the relevant Landau-Ginsburg potential. The fixed point of this Ginsburg-Landau potential is identified with the N=2 superconformal model. The latter superconformal model originates the topological conformal field theory that we are studing. According to ref.[15] the potential w is the quasihomogenius polinomial. It is possible to introduce the weights of the variables $x_1, ..., x_n$ in the way such that $w(x)$ is the quasihomogenius polynomial of degree one. Next we consider he polynomial $W(x, s)$ that is the minimal versal deformation of the polinomial $w(x) = W(x_1, ..., x_n; 0, ..., 0)$. This polinomial is the Landau-Ginsburg potential that corresponds to the perturbed topological field theory that we are studing. It has the form

$$W(x_1, ..., x_n; s_1, ... s_\mu) = w(x) + \sum s_j \phi_j(x). \tag{29}$$

Here $s_i$ are some coordinates on the coupling space. and the polinomials $\phi_i(x)$ form the basis of the local algebra of the corresponding singularity.

Let us define the coefficients $c^i_{jk}$ by the equations:

$$\frac{\partial W}{\partial s_j}\frac{\partial W}{\partial s_k} = \sum_i c^i_{jk}\frac{\partial W}{\partial s_i} \quad mod \quad (\partial_{x_1}W, ..., \partial_{x_n}W). \tag{30}$$

These coefficients form the associative algebra. Remark that $\phi_i(x) = \frac{\partial W}{\partial s_i}$.

Let us define the tensors $\eta_{ij}(s), c_{ijk}(s)$ by the equations

$$\eta_{ij} = res\frac{\phi_i\phi_j}{\partial_{x_1}W...\partial_{x_n}W}, \tag{31}$$

$$c_{ijk} = res\frac{\phi_i\phi_j\phi_k}{\partial_{x_1}W...\partial_{x_n}W}. \tag{32}$$

Here the function $res\ \psi(x,s)$ is defined as the integral

$$res(\phi) = \int_{\Delta(s)} \phi(x,s)dx_1...dx_n \tag{33}$$

where $\Delta(s) = \{(x_1,...,x_n) : |\partial_{x_k}W|^2 = \epsilon_k, k = 1,...,n\}$ and $\epsilon_k$ are arbitrary small numbers [17]. It is easy to see that the tensors $c^i_{jk}$, $c_{ijk}$, $\eta_{ij}$ from eqs. (31), (32), (30) are connected by the relation $c_{ijk} = \eta_{ip}c^p_{jk}$.

The remarkable fact is [14] that the tensors $\eta_{ij}$, $c_{ijk}$, defined by eqs. (31), (32), coincide with the tensors $\eta_{ij}$, $c_{ijk}$ defined by eqs. (17), (18).

This statement forms the bridge between the singularity theory and the topological conformal field theories. The physically rigorous proof of the statement is based on the evaluation the functional integral $< \prod \phi_i(x)exp(-S) >$ where S is the action of the topological field theory by the stationary phase method. The key step is to note that the latter functional integral is explicitly saturated by the trajectories in the space of fields $x_i(z)$ that are independent of z, i.e. it is equal to the finite dimensional integral over $x_i$. The explicit calculation gives eqs. (31) , (32) .

The metric (31) is nondegenerated and this is the standard fact [17,29]. Suiprisingly, this metric has the zero curvature [21,22]. Hence there are canonical coordinates in which this metric is constant. (These coordinates are defined up to the linear transformation with the constant coefficients). There are two proofs of this fact. One is the mathematical proof [21,22] . The second is the physicaly rigorous proof based on Ward identities of the topological field theory [11,12] and the identification of the metrics given by eqs. (17) and (31) . The physical proof gives the additional information. Namely in the coordinates, such that $\eta_{ij}$ is constant, the functions $c_{ijk}$ are the third derivatives of the function F (see eq. (19)), having the physical meaning of the partition function of the topological matter coupled to the two-dimensional topological gravity. This fact is apparently not known in the singularity theory. It would be interesting to determine the meaning of this function in the singularity theory.

The coordinates in which the metric $\eta_{ij}$ is constant are defined up to a linear transformation with the constant coefficients. We call these coordinates the flat coordinates.

It is clear that the problems of the determination of the three point correlation functions and the partition function F for the given model are solved once we know the flat coordinates. Indeed, once we know the expression of the Landau-Ginsburg potential $W(x, s(t))$ in the flat coordinates $t_i, i = 1, ..., \mu$, we can find all amplitudes using eqs. ( ¿¿2.13¡¡) and (30). The covariant derivatives in these coordinates are the usual ones. Consequently, F is determined by the direct integration of the three point couplings. The problem of solving the topological conformal field theory (and also two dimensional topological gravity theory) is the problem of how to find these flat coordinates.

## 3. THE FLAT COORDINATES AND THE GAUSS-MANIN EQUATIONS.

In this section we shall discuss the methods of the determination of the flat coordinates in the singularity theory.

This problem is connected with the theory of Gauss-Manin differential systems.

## 3.1. THE GAUSS-MANIN EQUATIONS.

Let $s_1$ be the coefficient in the superpotential (29) at the term that corresponds to the unit operator that is let $\phi_1 = 1$. Let $s' = (s_2, ..., s_\mu)$ and $s = (s', s_1)$. We define $\delta_W^\lambda$ as

$$\delta_W^\lambda = \frac{\Gamma(\lambda+1)}{2\pi i}(-W)^{-\lambda-1}. \tag{34}$$

Denote by $M_W^\lambda$ the free $R(x, s')$ module with the basis $(\delta_W^{(\lambda-k)})_{k\in Z}$ :

$$M_W^\lambda = \bigoplus_{k\in Z} R(x,s')\delta_W^{(\lambda-k)}. \tag{35}$$

Here R(x, $s'$) is the ring of all polynomials in $(x_1, ...x_n; s_2, ..., s_\mu)$.

Define the operators $D_{s_1}, s_1, D_{s_j}, D_{x_i}$ by the following formulae [27,30,28]:

$$\begin{aligned} D_{s_1}\delta_W^\lambda &= \delta_W^{\lambda+1}, \\ s_1\delta_W^\lambda &= -(W-s_1)\delta_W^\lambda - \lambda\delta_W^{(\lambda-1)}, \\ D_{s_j}\delta_W^\lambda &= \partial_{s_j}(W)\delta_W^{\lambda+1}, \\ D_{x_i}\delta_W^\lambda &= \partial_{x_i}(W)\delta_W^{\lambda+1}. \end{aligned} \tag{36}$$

Under these operations $M_W^\lambda$ becomes the left $D(x,s)[D_{s_1}^{-1}]$ module generated by the symbol $\delta_W^\lambda$. Here $D(x,s)$ is the algebra of all differential operators with the polynomial coefficients in $(x_1, ..., x_n; s)$.

The Gauss-Manin system is the system of equations on the integrals of the holomorphic differential forms over the vanishing cycles of the singularity. It is verified by the integrals of the form

$$u(s) = \int_{\gamma(s)} \phi(x,s)W^{-\lambda-1}dx \quad (dx = dx_1 \wedge .... \wedge dx_n). \tag{37}$$

Here $\phi(x,s)$ is a function, $\gamma(s) \subset \{(x_1,...,x_n)|W(x,s) \neq 0\}$ is any locally constant family of the n-dimensional cycles, lying in the neighbourhood of the point $(x=0, s=0)$. These cycles are called vanishing cycles (see ref. [17]).

Recall that $\phi_i(x)$ are the generators of the chiral ring associated with the given singularity. Let

$$w_i = \phi_i dx_1 \wedge .... \wedge dx_n. \tag{38}$$

Define

$$u_i = \int \delta_W^{(\lambda)} w_i \tag{39}$$

where $\delta_W$ is given by eq. (34) . Let $\vec{u} = (u_1, ...u_\mu)$. The Gauss-Manin system has the form

$$\begin{aligned} s_1\vec{u} &= A(s', D_{s_1})\vec{u}, \\ D_{s_k} D_{s_1}^{-1}\vec{u} &= B^{(k)}(s', D_{s_1})\vec{u}. \end{aligned} \tag{40}$$

Below we consider the case when $\lambda$ is zero.

The matrices A and $B^{(k)}$ in eq. (40) can be represented as the Loran series:

$$\begin{aligned} A &= \sum_{k=0}^{\infty} A_k(s') D_{s_1}^{-k}, \\ B^{(k)} &= \sum_{i=0}^{\infty} B_i^{(k)}(s') D_{s_1}^{-i}. \end{aligned} \tag{41}$$

The matrices $A$ and $B^k$ in eq. (3.10) can be easily computed (see e.g. [28] ). Let $w$ be any n-form. Then there exist the polinomials $a_1, ..., a_\mu$ in $D(s')$ such that

$$\int w\delta_W^\lambda = \sum_{i=1}^{\nu} a_i w_i + D_{s_1}^{-1} \int \delta_W^\lambda \xi. \tag{42}$$

Here $\xi = -d\eta$, where the form $\eta$ is the (n-1)-form defined by the equation

$$w = \sum_{i=1}^{\nu} a_i w_i + dW \wedge \eta. \tag{43}$$

We can repeat the above calculation for the form $\xi$, obtaining for the integrals of this form over the vanishing cycles the expansion of the type (42) .

The equation (43) gives the simple algorithm for calculating the matrices $A_i$ and $B_i^{(k)}$. Let $\vec{\phi} = (\phi_1, ..., \phi_\mu)$. Then

$$-W\vec{\phi} = A_0\vec{\phi} \quad mod\ (\partial_{x_i}W),$$
$$\partial_{s_i}(W)\vec{\phi} = B_i^{(k)}\vec{\phi} \quad mod(\partial_{x_i}W) \tag{44}$$

Note that the matrix $A_0$ defines the equation of the discriminant of the singularity [30]. The equation of the discriminant is

$$det(s_1 I - A_0(s')) = 0. \tag{45}$$

In order to find the matrix $A_1$ we must find the expansion:

$$W\phi_i = \sum_k (A_0)_{ik}\phi_k + \sum_k B_{ik}\partial_{x_k}W. \tag{46}$$

The matrix $A_1$ is defined from the equation

$$\sum_k \frac{\partial B_{ik}}{\partial x_k} = \sum_s (A_1)_{is}\phi_s \quad mod\, \partial_{x_k}W. \tag{47}$$

Continuing this procedure we can find all the matrices $A_i$ and $B_i^{(k)}$ Note that in the case of the minimal models there are at most two terms in the expansion (41). In the case of the models with d=1 there are at most 3 terms in these expansions.

It may be possible that the Gauss-Manin equations can be interpreted as the "Fourie-images" of the Ward identities. The functions $u_i$ can be considered as the "Fourie-images" of the primary fields $\phi_i$ of the chiral algebra.

The theory of the Gauss-Manin systems is closely connected to the theory of the primitive forms. In order to define the primitive forms we need the notion of the so called higher residue pairings [26,30,31]. The higher residue pairing puts into the correspondence to every two differential forms $w = \phi(x,s)dx_1...dx_n, \quad w' = \phi'(x,s)dx_1...dx_n$ and the function $W(x,s)$ the sequence of functions $K^{(k)}(w,w')$ dependent on $s'$. In this process we first build from the function W and the differential form $w$ the sequence of functions $\nabla^{(k)}(w)$ dependent on variables $(x,s)$. Next we put $K^{(k)}(w,w') = Res(\nabla^k(w)\phi')$. Namely, the higher residue pairings are defined in the following way. Consider the finite open covering of the space with the coordinates $(x_1,...,x_n;s)$ by the sets $U = \{U_i\}; \quad i = 1,...,N$. Here $U_i$ denotes the set $\{(x_i,s); \frac{\partial W}{\partial x_i} \neq 0\}$. Let us denote by $Q^{p,q}$ the following linear space. The elements of $Q^{p,q}$ are the sets of holomorphic q-forms. Namely, $w \in Q^{p,q}$ is the set of holomorphic q-forms $w^q(U_{i_1} \cap ... \cap U_{i_{p+1}})$ defined on each of the spaces $U_{i_1} \cap ... \cap U_{i_{p+1}}$: $w = \{w^q(U_{i_1} \cap ... \cap U_{i_{p+1}})\}$. The elements of $Q^{p,q}$ are called p-chains (see ref. [32] for the detailes). For example the elements of $Q^{0,q}$ are the sets of holomorphic q-forms defined on each of the sets $U_i$: $w \in Q^{0,q} = \{w^q(U_i); i = 1,...,N\}$. The elements of the space $Q^{1,q}$ are the sets of q-forms defined on each of the sets $U_i \cap U_j$, etc . We now define the coboundary operator $\partial$. The Chech coboundary operator acts as $\partial : Q^{p-1,q} \to Q^{p,q}$ : $w^q(U_{i_1}...U_{i_{p+1}}) = \sum(-1)^s w^q(U_{i_1} \cap ....\hat{U}_{i_s}... \cap U_{i_{p+1}})$. Here the sign $\hat{U}$ means that this particular set is omitted. This operator maps p-1-chains into p-chains. We now define the space $S^{p,q}$

$$S^{p,q} = Q^{p,q} \otimes [[\delta_0^{-1}]]. \tag{48}$$

Here $\delta_0$ is the formal parameter. The element of this space is the formal series

$$L = \sum_{i=-\infty}^{i=0} w_i \delta_0^{-i}. \tag{49}$$

here $w_i$ belong to the spaces $Q^{p,q}$ that were defined above. The coboundary operator $\partial$ acts as $S^{p,q} \to S^{p+1,q}$ on this complex. Define the cohomology operator $\hat{d}$:

$$\hat{d} = t_0^{-1} d - dW \wedge . \tag{50}$$

The cohomology operator $\hat{d}$ acts as $Q^{p,q} \to Q^{p,q+1}$. We can now define the double complex as the triple $(S^{p,q}, \partial, \hat{d})$. For every n-form $w$ we construct the element of $S^{0,n}$ by restricting this form on each of the sets of the covering. According to Saito there exists the function L$\subset S^{(n,0)}$ that is equal to

$$L = \hat{d}^{-1}(\partial \hat{d}^{-1})^n w. \tag{51}$$

We now define [26,30,31] the functions $\nabla^{(k)}(w)$ as the coefficients in the formal expansion

$$L = \sum_{k=0}^{\infty} \nabla^{(k)}(w) \delta_0^{-k}. \tag{52}$$

We define the higher residue pairings as the bilinear forms:

$$K^{(k)}(w, w') = Res[(\nabla^{(k)} w)\phi']. \tag{53}$$

For example, then

$$\begin{aligned} K^{(0)}(w, w') &= Res \frac{\phi' \phi}{\partial_{x_1} W ... \partial_{x_n} W}, \\ K^{(1)}(w, w') &= 1/2 \sum_{i=0}^{n} Res \frac{\frac{\partial \phi}{\partial x_i} \phi' - \phi \frac{\partial \phi'}{\partial x_i}}{\partial_{x_1} W ... (\partial_{x_i} W)^2 ... \partial_{x_n} W}. \end{aligned} \tag{54}$$

and so on. Note that $K^{(0)}$ is just the metric $\eta_{ij}$ defined above.

We conclude that the metric $\eta_{ij}$ is included into the infinite series of the bilinear forms. It would be interesting to check whether $K^{(i)}$ have any connection with the higher genus corrections for the metric on the space of all couplings ( see e.g. refs. [1,33,12] for the discussion of the higher genus corrections to the correlation functions).

## 3.3. GAUSS-MANIN SYSTEMS AND FLAT COORDINATES

We now consider the connection between the Gauss-Manin systems and flat coordinates [21,22]. According to refs. [21,22] the flat coordinates can be found in two steps. First, we go from the basis $(u_1, ..., u_\mu)$ in Gauss-Manin equations to the basis $(v_1, ..., v_\mu)$ in which the Gauss-Manin equations have the simple form (see below). Second, we look for the coordinates $(t_1, ..., t_\mu)$ on the coupling space for which $v_i = \int \frac{\partial W(x,s(t))}{\partial t_i} \delta(W) dx_1 \wedge ... \wedge dx_n$. It appears that such coordinates are flat [21,22].

In more details, it was proved in [21,22] that there exist a special basis on the space of differential forms $\vec{v}$ in which the Gauss-Manin system acquires the following simple form

$$\begin{aligned} s_1 \vec{v} &= (A_0(s') + A_1(s') D_{s_1}^{-1}) \vec{v}, \\ D_{s_k} D_{s_1}^{-1} \vec{v} &= B(s')^{(k)} \vec{v}. \end{aligned} \tag{55}$$

Here $A_1$ is the diagonal matrix whose elements are the U(1) charges of the basic chiral fields $\phi_i$ and $A_0$ is nilpotent. For this basis the higher residue pairings defined in the previous section are zero:

$$K^{(i)}(v_i, v_j) = 0, \; i \geq 1 \tag{56}$$

and the metric $\eta$ is constant: $\eta_{ij} = K^{(0)}(v_i, v_j)$ =const.

The basis that we are looking for is connected to the basis $\vec{u}$ in which we wrote the eqs. (41) by the transformation:

$$\vec{u} = S(s', D_{s_1}^{-1}) \vec{v}. \tag{57}$$

Here the matrix S is the holomorphic function of its variables. In order to find S we must make the Fourie transformation over the variable $s_1$ in eqs. (3.16). We denote the Fourie image of $s_1$ as x and the Fourie images of $\vec{u}$ as $\vec{F}$ and of $\vec{v}$ as $\vec{G}$:

$$F = SG. \tag{58}$$

We shall denote the Fourie image of the matrix S also as S. The connection (55) acquires the following form after the Fourie transformation:

$$D_xF = M\vec{F} = (M_0/x^2 + M_1/x + M_0 + ....)F,$$
$$D_{t_i}F = N^{(i)}\vec{F} = (N_0^{(i)}/x + N_0^{(i)} + N_1^{(i)}x + ...)\vec{F}. \tag{59}$$

The Fourie image of the matrix S transforms the connection (59) into the form

$$D_xG = M'\vec{G} = (M_0/x^2 + M_1/x)G,$$
$$D_{t_i}F = N'^{(i)}\vec{G} = (N_0^{(i)}/x)\vec{G}. \tag{60}$$

Here the matrices $M_i, M_i'$, $N_i, N_i'$ are independent of x. Under the change of the basis given by eq. (58) the system (59) transforms as

$$M' = S^{-1}MS + S^{-1}\partial_x S \tag{61}$$

or equivalently

$$SM' = MS + \partial_x S. \tag{62}$$

The analogous transformation can be written for the matrix N. The eq. (62) gives the system of recurrence relations for the coefficients in the expansion:

$$S = \sum S_i x^i. \tag{63}$$

Hence we can determine the matrix S. In general there is an infinite number of terms in the r.h.s. part of eq. (63) .

Once we determine the matrix S we can find the basis v for which eqs. (59) are valid. In the same way we can determine the matrix $S^{-1}$. Once we know $S^{-1}$ we can find the set of forms $e_i$ such that

$$v_i = \int e_i \delta(W) dx \tag{64}$$

Next we use the fact that $D_{s_1}^{-k}u$ can be expressed as the linear combination $\sum F_i(s_i, s_1)u_i$, where $u_i$ are integrals of the differential forms that form the basis in the local algebra (chiral ring) of the singularity. Hence there exist the matrix $Q(s_1, ..., s_\mu)$ such that $\vec{v} = Q\vec{u}$.

It was noted in [22] that there exist the coordinates on the space of the parameters of the versal deformation such that

$$\frac{\partial W}{\partial t_i} = e_i. \tag{65}$$

Moreover, this coordinate system is flat. We see that in order to find the flat coordinates we must find the ccoordinates $t_i$ that put the Gauss-Manin system associated with the given singularity into the simple form (55). In order to achieve this goal we have to solve the eqs. (60) and find the matrix S of the basis transformation. Next we have to find the flat basis $v_i$ solving the eq. $\vec{u} = S\vec{v}$. Finally we find the flat coordinates using eq. (65). Constructively, this method permits the simple determination of flat coordinates if S is a finite polynomial in $D_{s_1}^{-1}$. This is the case of the minimal models as we shall see later. In general however, starting from the d=1 case S is the infinite series in $D_{s_1}^{-1}$ and this procedure gives only the way to find the series expansions of the flat coordinates and permit the one to prove their existence.

### 3.4. PRIMITIVE FORMS

Let us denote by $\Upsilon$ the space of all vector field in variables $s'$. The primitive form [21,30] is the form for which the following axioms are fulfilled:

$$\begin{aligned}
&1) K^{(1)}(\nabla_\delta \xi^{(-1)}, \nabla_{\delta'} \xi^{(-1)}) = 0,\\
&2) \nabla_E \xi^{(0)} = (r-1)\xi^{(0)},\\
&3) K^{(k)}(\nabla_\delta \nabla_{\delta'} \xi^{-2}, \nabla_{\delta''} \xi^{(-1)}) = 0 \quad k \geq 2 \, \forall \delta, \delta', \delta'' \in \Upsilon,\\
&4) K^{(k)}(s_1 \nabla_\delta \xi^{(-1)}, \nabla_{\delta'} \xi^{(-1)}) = 0 \quad k \geq 2 \, \forall \delta, \delta'.
\end{aligned} \tag{66}$$

Here $\xi^{(0)}$ is the primitive form; $\xi^{(-k)} = (\nabla_{\delta_1}^{-k})\xi^{(0)}$. The symbol $\nabla$ denotes the Gauss-Manin connection (see refs. [17,18,19] for the detailed definition; this connection must not be mixed with the connection $\nabla$ associated with the metric (17)). The vector field E is the Euler vector field. This is the field such that $\nabla_E + I$ acts as the U(1) current. The number r is the sum of the U(1) charges of the fields $x_1, ..., x_n$. For simple singularities $W(x,s) = x^\mu + ....$ we have $\xi^{(0)} = dx^{30}$. We can also consider the example of the simple elliptic singularities [21]

$$
\begin{aligned}
&yz^2 - x(x-y)(x-\lambda y) + ....., \\
&xy(x-y)(x-\lambda y) + z^2 + ...., \\
&x(x-y^2)(x-\lambda y^2) + z^2 + ....
\end{aligned}
\tag{67}
$$

etc. Here the sign ... means the terms corresponding to the minimal versal deformation. Let $w = dx \wedge dy \wedge dz$. Then the primitive form is

$$
\xi^{(0)} = w/(c\int_{\gamma_1} res_E(w) + d\int_{\gamma_2} res_E(w)). \tag{68}
$$

Here $(c,d) \neq (0,0)$, $\gamma_i$ are the basis of the horisontal family of the homology of the corresponding elliptic curves E. For the detailed investigation of the definition and properties of the primitive form see refs. [21,22].

Let us define $v_1$ as the integral over the vanishing cycles of the monomial with the minimal weight in the theory (that corresponds to the unit operator in the conformal field theory language). Suppose $\xi$ is the integral of the primitive form $\xi^{(0)}$ over the vanishing cycles.

$$
\xi = D_1^{-1}\int \xi^{(0)}\delta(W)dx \tag{69}
$$

It was proven in ref. [22] that

$$
\partial_i \xi = v_i \tag{70}
$$

We differentiate here in the flat coordinate basis. The basis $\vec{v}$ is the basis in which the Gauss Manin system has the simple form. This basis was defined in the last section.

Let us define the following differential form on the space of couplings:

$$
w = \sum_{i=1}^{\nu} K^{(0)}(\frac{\partial W}{\partial_{s_i}}\xi^{(0)}, \xi^{(0)})ds_i. \tag{71}
$$

It turns out that $w = d\tau$ and $\tau$ is the flat coordinate at the operator that corresponds to the operator with the highest U(1) charge. For example, for the case of the elliptic singularities with d=1 the flat coordinate has the form [21]

$$\tau = (a\tau_0 + b)/(c\tau_0 + d); \quad a, b, c, d \in C; ad - b \neq 0. \tag{72}$$

$$\tau_0 = \int_{\gamma_1} res_E(w) / \int_{\gamma_1} res_E(w) \quad \gamma_1, \gamma_2 \in H_1(E, Z). \tag{73}$$

It is interesting to know whether the other flat coordinates satisfy the similar relations.

It is also worth mentioned that we can define the flat coordinates as the independent nonconstant solutions of the system of the differential equations [26]:

$$(\delta\delta' - \nabla_\delta \delta')t = 0 \ \forall \delta, \delta' \in \Upsilon. \tag{74}$$

## 3.5. FLAT COORDINATES AND MONODROMY GROUPS.

The problem of finding the flat coordinates arises in the singularity theory also from the point of view of the problem of finding the special basis for the ring of the functions, invariant under the monodromy group. This problem was considered for the case of the simple singularities [23,24,25] and for the case of the elliptic singularities [26].

Let W be the finite group generated by reflections and acting on the $\mu$-dimensional vector space V. This group acts on the space of polynomials S on the vector space. The subring $S^W$ of the invariant polinomials is generated by $\mu$ algebraically independent homogenius polynomials. Irreducible finite groups of reflections are classified by their types: $A_\mu, D_\mu, E_6, E_7, E_8$.... In the singularity theory the classification of the critical points of functions also begins with the types $A_\mu, D_\mu, E_6, E_7, E_8$.... In this situation the space V, the group W, the W-invariant scalar product I on the space V and the ring $S^W$ are identified with the following objects for the corresponding singularity. $V$ is the space of the vanishing homologies, I is the intersection form in homologies, W is the monodromy group of the corresponding singularity, $S^W$ is the ring of the polynomials on the parameter space of the minimal versal deformation (see ref. [17]).

Let us define the operation of the contraction of the invariants

$$< dP, dQ >= \sum_{i,j=1}^{l} \frac{\partial P}{\partial \chi_i} \frac{\partial Q}{\partial \chi_j} I(\chi_i, \chi_j). \tag{75}$$

Here $\chi_i$ are the coordinates of V. This operation puts into correspondence to every two W-invariant functions P and Q the W-invariant function $< dP, dQ >$. Let us define the matrix function on the space of the parameters of the versal deformation $< dP_i, dP_j >$ where $P_1, ..., P_\mu$ is the basis of the ring $S^W$ . Let $P_\mu$ be the element of the basis $(P_1, ..., P_\mu)$ that has the maximum power of the homogenuity. The authors of refs. [23,24,25] proved the existence of the basis such that the matrix $\frac{\partial}{\partial P_\mu} < dP_i, dP_j >$ is constant. Moreover this basis is the flat coordinate basis in the sence discussed in the present paper.

The analogous investigation was made in the case of the elliptic singularities that correspond to the case d=1 [26]. In this paper the polinomials $(P_1, ..., P_\mu)$ are changed to the theta functions.

We saw the direct connection between the theory of Gauss-Manin systems, K. and M. Saito theory of the primitive form, monodromy groups and flat coordinates. This connection works in the most effective way for the case of the minimal models.

It would be interesing to find whether the metric of contraction of the invariants given by eq. (75) and the closely related to it symplectic structure on the space of couplings (see e.g. ref. [17]) are of any physical significance.

## 4. MINIMAL MODELS.

In this section we shall explain how the technique discussed above works in the case of the minimal models. In fact, all necessary calculations were really done by Japanese mathematicians in refs. [23,24,25,28,32]. So we'll limit ourselves just by explaining the simple technique connected with the Gauss-Manin systems and proposed in [28]. As we shall see, this technique gives the full solution of the minimal topological theories including the previously unknown cases of $E_7$ and $E_8$ models. For the known case of the other minimal models our results of course coincide with the results of refs. [11,12].

Let us explain the method [28]. We consider the following series of singularities:

$$\begin{aligned} &W = x^{l+1} + t_{l-1}x^{l-1} + .... + t_0 : \quad A_l \\ &W = x^{l-1} + 1/2xy^2 + t_{l-3}x^{l-3} + ... + t_0 : \quad D_l \\ &W = x^4 + y^3 + t_0 + t_1 x + t_2 x^2 + t_3 y + t_4 xy + t_5 x^2 y : \quad E_6 \\ &W = x^3 y + y^3 + t_0 + t_1 y + t_2 xy + t_3 xy^2 + t_4 x + t_5 x^2 + t_6 y^2 : \quad E_7 \\ &W = x^3 + y^2 + t_0 + t_1 x + t_2 y + t_3 x^2 + t_4 x^3 + t_5 xy + t_6 x^2 y + t_7 x^3 y : \quad E_8 \end{aligned} \tag{76}$$

Note now that in the case of the minimal models the matrices A, $B^{(i)}$ in the Gauss -Manin system have the form

$$\begin{aligned} A &= \sum_{k=0}^{1} A_k(t) D_0^{-k}, \\ B^{(k)} &= \sum_{i=0}^{1} B_i^{(k)}(t) D_0^{-i}. \end{aligned} \tag{77}$$

Indeed, deg $(We_i) \leq 2l$. Hence deg $B_{ik} \leq l$ and deg $\sum_k \frac{\partial B_{ik}}{\partial x_k} \leq l-1$ (see eq. (47)). Hence, according to the algorithm in sec. 3 $A_r, B_r^{(k)} = 0$ for $r \geq 2$. (since $\sum \frac{\partial B_{ik}}{\partial x_k}$ is the linear combination of the basic monoms without the $\partial_{x_i} W$ terms in the r.h.s.) If we substitute the matrix S into such system we immmediately see that $S = S_0$, i.e. S is the independent of $D_0^{-1}$. The matrix S satisfies the system of differential equations:

$$\frac{\partial S}{\partial t_k} = B_1^{(k)} W. \tag{78}$$

The flat coordinates $s_i$ are the solution of the system of the differential equations

$$\frac{\partial s_i}{\partial t_k} = S_{ki}. \tag{79}$$

This system was solved in ref. [28] where the following result for the flat coordinates was obtained:

$$s_\mu = t_0 \delta_{\mu,0} + \sum_{<\sigma,\alpha>=\sigma_\mu} c_\mu(l(\alpha)) \frac{t^\alpha}{\alpha!}. \tag{80}$$

The matrix S is diven by the following formula:

$$S_{ij} = \sum_{<\sigma,\alpha>=\sigma_i-\sigma_j} c_i(l(\alpha)+j) \frac{t^\alpha}{\alpha!}. \tag{81}$$

The function $c_\mu$ is defined by the following set of equations:

1) Case of $A_l$: Define $L(\mu) = (\alpha \in N; \alpha \equiv \mu(l+1) = \{\mu + k(l+1); k \geq 0\}$,

$c_\mu(\alpha) = (-1)^k (\frac{\mu+1}{l+1}; k)$ if $\alpha \in L(\mu)$,

$c_\mu(\alpha) = 0$ otherwise.

$l(\alpha) = \sum_{\mu=0}^{l-1} \mu\alpha_\mu \in N \quad \{N = 0, 1, ..., l-1\}$. Here

$$\frac{t^\alpha}{\alpha!} = \frac{t_1^{\alpha_1} ... t_{\alpha_1 !}^{\alpha_{l-1}}}{\alpha_1 ! ... \alpha_{l-1}!}. \tag{82}$$

The symbol $(x; k)$ denotes

$$(x; k) = x(x+1)...(x+k-1) \tag{83}$$

2) Case of $D_l$. Take $N = \{(\mu, 0); 0 \leq \mu \leq l-2\} \cup \{0, 1\}$.

$L(\mu, 0) = \{(\alpha_1, \alpha_2) \in N^2; \alpha_2 \equiv 0 \quad mod \quad 2, \alpha_1 = \mu + \frac{\alpha_1}{2} \quad mod \quad (l-1)\}$,

$= \{(\mu + (k_1(l-1) + k_2, 2k_2); k_2 \geq 0, k_1 \geq -\frac{\mu+k_2}{l-2}\}$,

$L(0, 1) = \{(\alpha_1, \alpha_2) \in N^2; \alpha_2 \equiv 1 \, mod \, 2, \alpha_2 = \frac{\alpha_1 - 1}{2} \, mod \, (l-1)$,

$= \{(k_1(l-1) + k_2, 2k_2 + 1); k_2 \geq 0, k_1 \geq \frac{-k_2}{l-1}\}$,

$c_{\mu 0}(\alpha) = (-1)^{k_1+k_2}(\frac{\mu+1}{l-1} - \frac{1}{2(l-1)}; k_1)(1/2; k_2)$ if $\alpha \in L(\mu; 0)$,

$c_{\mu,0}(\alpha) = 0$ otherwise.

$c_{01}(\alpha) = \frac{(-1)^{k_2} k_2!}{(-k_1)!}$ if $\alpha \in L(0,1), k_1 \leq 0$,

$c_{01} = 0$. Here $l(\alpha) = (l_1(\alpha), l_2(\alpha))$, $l_i(\alpha) = \sum_{\mu \in N} \mu_i \alpha_\mu$. The function $l(\alpha)$ is defined in the same way also for three other exeptional singularities.

3) Case of $E_6$. $N = (\mu_1, \mu_2); \mu_1 = 0, 1; \mu_2 = 0, 1$.

$L(\mu_1, \mu_2) = \{(\alpha_1, \alpha_2) \in N^2; \alpha_1 \equiv \mu_1 \, mod \, 4, \alpha_2 \equiv \mu_2 \, mod \, 3\}$,

$= \{(\mu_1 + 4k_1, \mu_2 + 3k_2); k_1, k_2 \geq 0\}$,

$c_\mu(\alpha) = (-1)^{k_1+k_2}(\frac{\mu_1+1}{4}; k_1)(\frac{\mu_2+1}{3}; k_2)$ if $\alpha \in L(\mu)$,

$c_\mu(\alpha) = 0$ otherwise.

4) Case of $E_7$. $N = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0)\}$,

$L(\mu_1, \mu_2) = \{(\alpha_1, \alpha_2) \in N^2; \alpha_1 \equiv \mu_1 \, mod \, 3, \alpha_2 \equiv \mu_2 + (\alpha_1 - \mu_1)/3 \, mod \, 3\}$,

$= \{(\mu_1 + 3k_1, \mu_2 + 3k_2 + k_1); k_1 \geq 0, k_2 \geq -(\mu_2 + k_1)/3$,

$c_\mu(\alpha) = (-1)^{k_1+k_2}((\frac{\mu_1+1}{3}); k_1)((\mu_2+1)/3 - (\mu_1+1)/9; k_2) \quad if \, \alpha \in L(\mu)$,

$c_\mu(\alpha) = 0$ otherwise.

5) Case of $E_8$. $N = \{(\mu_1, \mu_2); \mu_1 = 0, 1, 2, 3; \mu_2 = 0, 1\}$,

$L(\mu_1, \mu_2) = \{(\alpha_1; \alpha_2) \in N^2; \alpha_1 \equiv \mu_1 \, mod \, 5, \alpha_2 \equiv \mu_2 \, mod \, 3\}$.

$= \{(\mu_1 + 5k_1, \mu_2 + 3k_2); k_1, k_2 \geq 0\}$.

$c_\mu(\alpha) = (-1)^{k_1+k_2}((\mu_1+1)/5; k_1)((\mu_2+1)/3; k_2)$ if $\alpha \in L(\mu)$,

$= 0$ otherwise.

In these formulae the mapping $l(\alpha)$ from $N^l$ to $N^2$ is defined in the following way. Put

$$W(x, y) = \sum_k t_k x^{\mu_1(k)} y^{\mu_2(k)}. \tag{84}$$

We define

$$l_1(\alpha) = \sum_k \alpha_k \mu_1(k), \; l_2(\alpha) = \sum \alpha_k \mu_2(k). \tag{85}$$

The scalar product $< \sigma, \alpha >$ is given by the sum $\sum_i \sigma_i \alpha_i$. Here

$$\sigma_k = 1 - \sum_i \rho_i \mu_i(k). \tag{86}$$

The parameters $\rho_1$ and $\rho_2$ are the weights of the generators x and y of the polynomial algebra (equal to the U(1) charges of the corresponding primary fields).

It is possible to check that in the cases of $A_\mu, D_\mu$, $E_6$ singularities we obtain the results of refs.[11,12]. We have also presented the explicit formula for the flat coordinates in the cases of $E_7$ and $E_8$ models.

Once the flat coordinates are known we can use the eqs. (30) in order to calculate the structure constants $c^i_{jk}$. The correlation functions $c_{ijk}$ are obtained by raising and lowering the index i using the flat metric whose entries are constants. The partition function is obtained by the trivial three time integration of $c_{ijk}$. Note that using the recursion relations and the factorisation properties of the correlation functions it is straightforward to calculate all other physical amplitudes of the model at all genuses once we know the structure constants $c_{ijk}$ [4,5,8,9,10,11,12] .

## 5. CONCLUSION.

The important feature of the technique disussed above is that it is valid in the case of the arbitrary topological field theories. Indeed, all theorems concerning the connection between the topological conformal field theories and the singularity theory that were discussed in sec. 3, 4 are valid independently of the value of the central charge c. However, while the existence of the flat coordinate basis is still garanteed, the real calculations in the case $d \geq 1$ become much more complicated. The reason is that the number of terms in the presentation of the Gauss-Manin system in (41) can be now bigger than two. Hence the coefficients $S_i$ in the matrix S of eq. (57) are not zero any longer for $i > 1$. Instead we have a set of the recursion relations that express them one through another. Hence, the simple equations, like eq. (79), are not valid any more. For example, if we take the potential

$$W = x^3 + y^3 + z^3 + t_0 + t_1 x + t_2 y + t_3 z + t_4 xy + t_5 xz + t_6 zy + t_7 xyz, \tag{87}$$

then for this potential the matrix $B_2^{(7)} \neq 0$. Hence we get the complicated set of recursion relations and differential equations. Partial characterisation of the flat coordinates for the d=1 theories in terms of the theta functions is discussed in ref.[26]. Note also that strictly speaking we consider only the small vicinity of zero for the space of the parameters of the versal deformation. It is an open question (especially in the case $d > 1$) whether the theory can be defined globally on the space of couplings.

We conclude that the singularity theory gives the way to solve explicitly all topological conformal field theories at genus zero. The procedure is greatly simplified in the case of the minimal models (see section 4). As a final result we can get the partition function F of the topological matter coupled to the topological gravity.

We have seen the close connection between topological field theories and the theory of Gauss-Manin systems. The latter is the theory of the integrals of the differential forms over the vanishing cycles. It would be interesting to understand whether these Gauss-Manin systems can be interpreted as the "fourie transformed" Ward identitities of the topological field theory. At least in the mathematical proofs of the properties of the metric $\eta_{ij}$ this system seems to play the same role as the Ward identities in the physical proofs of the properties of this metric in [11,12]. It would be also interesting to find the mathematical meaning of the function F. It seems that this function was not considered yet in the singularity theory.

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