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Wormholes Made of Fermions*

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We consider quantum mechanics of gravity interacting only with massive or massless fermions. We find wormhole solutions carrying even fermion numbers. The size of wormholes is proportional to $Q^{2/3}$ (Q) when charge Q is small (large). Some of their consequences are discussed.

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1. Introduction

A wormhole considered here is a euclidean field configuration in some field theory containing gravity, consisting of two asymptotically flat regions connected by a tube, or throat. Wormholes are not mere euclidean configurations but solutions of the euclidean field equations. Typically, wormholes come in many types, distinguished by their sizes (radii at midpoint), the values of conserved global charges flowing down their throats, etc.

Wormholes can be interpreted as instantons in a given flat region, interpolating states of different charges. Their effect is to shift the cosmological constant and to generate some effective local lagrangian density at a scale lower than the wormhole scale [1]. This effective lagrangian depends on α parameters and in general violates global charge. Wormholes in short introduce an α-dependent shift in all couplings. By using wormhole physics and euclidean quantum gravity, Coleman [2] has introduced an argument to solve the cosmological constant. Here, the α parameters are fixed by two conditions: First, the cosmological constant should vanish. (See also Ref. [3].) Second, on the subset defined by the zero cosmological constant the Planck mass must be maximized [4]. This is supposed to fix all α's and so all coupling constants.

Giddings and Strominger's wormhole solution [5] for a theory of the antisymmetric tensor field coupled to gravity exists because the euclidean energy momentum tensor of the antisymmetric tensor field has the opposite sign from that of an ordinary scalar field. Later, one of us [6] argued that wormhole solutions can exist even for Goldstone bosons. In this case wormholes are considered in the context of quantum tunneling and conserved charges play a crucial role. These wormholes are equivalent to those for the antisymmetric tensor field. (See other aspects in Ref. [7].) Subsequently, it is shown by Abbott and M. Wise, [8], and Coleman and Lee [9] that wormholes can exist even when the global symmetry is not spontaneously broken. However, all these wormholes are based on the scalar fields [1] coupled to gravity with conserved global charge.

In this paper we reconsider wormholes in the context of tunneling and show that wormhole solutions can exist in a theory of a fermion field coupled to gravity. Conserved charge, in this case fermion number, plays again the crucial role.

[1] Recently, Hosoya and Ogura [10] have shown that there is a wormhole solution when there is a non-zero magnetic field in the theory of the nonabelian gauge field coupled to gravity. As the magnetic field in this theory is not gauge invariant, it would be interesting to investigate further properties of this wormhole solution.

This solves at least one obvious contradiction: In the zero quark mass limit pions are Goldstone bosons and give rise to wormholes carrying the axial $SU(2)$ chirality. However the wormhole scale, a geometric mean of the QCD and Planck scales, is much higher than the chiral symmetry breaking scale and in that scale pions do not exist. The configuration of wormholes carrying chirality should change from pionic to fermionic as the distance from the wormhole neck becomes smaller than the chiral symmetry breaking scale.

First, we relate the barrier penetration in the minisuperspace model of quantum cosmology with wormhole solutions. From this one can obtain wormhole solutions more directly. Then, we generalize this consideration to include fermions and show that wormholes carrying fermion number can exist. Concluding remarks will then follow.

2. Wormholes and tunneling

In the theory of the scalar field coupled to gravity, wormhole solutions can be obtained as $O(4)$ symmetric solutions of the Euclidean field equations with due attention to conserved charges. There is a somewhat different and simple way to get these wormhole solutions. This can be done by considering the minisuperspace model of quantum cosmology. (See for an example Ref. [11].)

The minisuperspace model is described by the metric $ds^2 = l_p^2[-dt^2+R(t)^2 d\Omega_3^2]$, where $l_p^2 \equiv 2/3\pi m_p^2$ and $d\Omega_3^2$ is the metric on the unit three sphere. (In this paper the scale of dimensional quantities is fixed by l_p.) We assume that matter is uniformly distributed on the three sphere. The Hilbert-Gibbons-Hawking action with the cosmological constant λ is

$$S_M = \int dt\, \frac{1}{2}[-RR'^2 + R - H^2R^3] \tag{2.1}$$

where $H^2 \equiv 4\pi^2 l_p^4 \lambda$. The field equation is

$$R'^2 - H^2R^2 + 1 = 0 \tag{2.2}$$

where the prime means d/dt.

Canonical momentum for $R(t)$ is then $p = -RR'$. Hamiltonian $\mathcal{H}$ is given by $-2R\mathcal{H} = p^2 + R^2 - H^2R^4$. After canonical quantization, $p = -i\partial/\partial R$, the dynamics is governed by the Wheeler-DeWitt equation, $\mathcal{H}\Psi(R) = 0$. This equation is basically the Schrodinger equation with zero energy. The reason for zero energy is that general relativity is a covariant theory and the wave function should not depend on the time coordinate. With additional matter fields, the effective potential $U(R)$ is defined so that $-2R\mathcal{H} = p^2 + U(R)$.

Suppose now that there is a Goldstone boson field $\theta(t)$ with the symmetry breaking scale v/l_p. The conserved $U(1)$ charge, Q, in the three sphere is $Q = 2\pi^2 R(t)^3 v^2 \dot{\theta}(t)$. With additional energy density due to this charge, the effective potential becomes

$$U(R) = R^2 - R_0^4/R^2 - H^2 R^4 \tag{2.3}$$

with $R_0^4 \equiv Q^2/2\pi^2 v^2$. In the minisuperspace model, the Goldstone boson field depends only on time and is represented by the total conserved charge [2]. A similar case occurs in two dimensional quantum mechanics when one uses the effective potential of states in a given angular momentum for the tunneling problem along the radial direction.

For a small λ, there are always two turning points, $R_s < R_l$, in the effective potential. The classically forbidden region is $R_s < R < R_l$. With $R \leq R_s$ a small universe carrying charge Q expands from zero to the turning point and contracts. This evolution is similar to that of a matter dominated closed universe. With $R \geq R_l$ the large universe is more like a DeSitter spacetime; it exponentially contracts to the turning point R_l and expands forever.

The Wheeler-DeWitt equation is

$$[-\frac{d^2}{dR^2} + R^2 - \frac{R_0^4}{R^2} - H^2 R^4]\Psi(R) = 0 \tag{2.4}$$

The standard WKB method for the wave function in the forbidden region leads that $\Psi(R) \approx \exp(\pm \int |p| dR)$, where $|p| = \sqrt{U(R)}$ is given by $\mathcal{H} = 0$ in the classical level. The momentum is imaginary in the forbidden region. We are interested in the minimum value of the exponent. The Maupertuis' variational principle in classical mechanics shows that the stationary path and value of $-2\int_{R_s}^{R_l} |p| dR$ with zero energy are identical to those of the euclidean action.

The euclidean action is

$$S_E = \int d\tau \frac{1}{2}[-R\dot{R}^2 - R + \frac{R_0^4}{R^3} + H^2 R^3] \tag{2.5}$$

[2] As energy and charge of the Goldstone boson depend on the time derivative of the field, the euclidean time form of them is not straightforward. Either we go through an elaborate argument [6][9] or we can represent the energy density in terms of charge before going to euclidean time. This procedure should be distinguished from the naive euclidean field theory, which can be correct or wrong depending on the situation.

where the dot means $d/d\tau$. A euclidean solution describes the tunneling from a small universe to a large universe, and vice versa. The field equation in euclidean time is

$$\dot{R}^2 + \frac{R_0^4}{R^4} + H^2 R^2 - 1 = 0 \tag{2.6}$$

In the case $Q = 0$, the solution, $R = H^{-1}\sin(H\tau)$, is for the metric of the euclidean four sphere, or that of euclidean DeSitter space, and $S_E = -4/3H^2$.

Let us consider the point of view of the small universe. In classical dynamics, we know that the universe will expand and contract when it hits the barrier. In quantum gravity, it is not clear at all. There are several unanswered questions in quantum gravity: (1) What is the probability function? (2) Is the quantum tunneling suppressed [3], or not? The answer for this question is also related to our future. Suppose that our universe is a closed three sphere and matter dominated. The classical scenario is that our universe will expand and contract. The quantum scenario can be very different; rather than contracting, the universe could expand forever even though it will be under the barrier in the later period. What kind of life will be in that later period?

Suppose that tunneling is suppressed. The 'tunneling rate' is proportional to $\exp(S_E)$. S_E goes to the negative infinity as the cosmological constant goes to zero. This means that it is very hard for a small universe to tunnel into a large universe, and vice versa. This is intuitively appealing.

However, there is an additional interpretation of this solution [4]. We take the euclidean configuration as a wormhole solution which connects one euclidean DeSitter spacetime with another euclidean DeSitter spacetime. Its interpretation seems to be more clear in the zero cosmological constant limit. In this case, this solution connects two flat euclidean

[3] As the wave function of the universe does not depend on the external time explicitly, the scale factor R plays the role of time. The Wheeler-DeWitt equation is in some sense a Klein-Gordon equation. The barrier we are talking about is not then an energy barrier, rather a time dependent mass term for the wave function of the universe. Hence, the quantum mechanical suppression of tunneling can not be blindly generalized to quantum gravity.

[4] There is one more interpretation. In this case, the wormhole throat is cut in half and is called a baby universe. The solution then interpolates between one flat universe and one flat universe plus a baby universe. The baby universe will contract in real time. As the solution has the characteristic of both instanton and bounce, this interpretation is controversial but has been useful in introducing the effective local action.

spacetimes. In a given universe, there is a classical degeneracy in the vacuum. The vacua can be classified by their total global charges, though their energies are identically zero. Wormhole solutions interpolate between these degenerate vacua by channeling out the charge differences into other universes.

For a wormhole solution, the action of the background universe, say that of the euclidean DeSitter spacetime, should be subtracted from (2.5). The action for the background spacetime is also negative and the difference S_w, which we will call the wormhole action, is finite positive, $S_w = R_0^2$. The exponential suppressing factor for wormhole instantons in the effective action is now proportional to $\exp(-S_w/2)$.

We will use the second interpretation for the fermionic wormholes.

3. Wormholes made of Fermions

Let us consider the fermion field lagrangian in curved spacetime background with a metric $g_{\mu\nu}$. To define a spinor, one needs a local frame form, $e^a_\mu dx^\mu$, such that $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$ and the gamma matrices γ^a such that $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$, where $\eta_{ab} = diag(-1,1,1,1)$. There is an induced spin connection ω^{ab}_μ, which is antisymmetric in ab and satisfies $\partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu} e^a_\rho + \omega^{ab}_\mu e_{b\nu} = 0$. The fermion action, which is invariant under the local coordinate and lorenz transformations, is

$$S = \int dx^4 \sqrt{-g}[\frac{1}{2}\bar{\psi}\gamma^a e^\mu_a D_\mu \psi + h.c. + m\bar{\psi}\psi] \tag{3.1}$$

where $D_\mu = \partial_\mu - \frac{i}{4}w_{\mu bc}\sigma^{bc}$, $\bar{\psi} = \psi^\dagger \eta$ and η can be chosen as $i\gamma^0$. There are conserved energy momentum tensor and fermion current:

$$T_{\mu\nu} = -\frac{1}{4}[\bar{\psi}\gamma^a e_{a\mu} D_\nu \psi + (\mu \leftrightarrow \nu)] + \frac{1}{2} g_{\mu\nu}[\bar{\psi}\gamma^a e^\rho_a D_\rho \psi + m\bar{\psi}\psi] + h.c. \tag{3.2}$$

$$J^\mu = \sqrt{-g}\bar{\psi}\gamma^a e^\mu_a \psi \tag{3.3}$$

The conservation law is $D_\mu T^{\mu\nu} = 0$ and $\partial_\mu J^\mu = 0$. Note that the current J^μ is a vector density.

On the minisuperspace background the energy density $\rho_F = -T^0_0$ is

$$\rho_F = \frac{1}{R}\bar{\psi}\not{D}_3\psi + m\bar{\psi}\psi \tag{3.4}$$

where $\not{D}_3 = \gamma^i e^m_i(\partial_m - \frac{i}{4}w^{jk}_m \sigma_{jk})$ with $i,j,k,m = 1,2,3$ is the part of the four-dimensional dirac operator on the unit three sphere. (This can be split into two three-dimensional dirac operators of mass $\pm m$ by the eigevalue ± 1 of $i\gamma^0\gamma^5$.) The total conserved charge is

$$Q = R^3 \int dV_3 \psi^\dagger \psi \tag{3.5}$$

where the integration is over the unit three sphere.

The symmetry of a unit three sphere is $SU(2) \times SU(2)$. The eigenvalues and eigenvectors of the symmetry and dirac operators on S^3 are known [12]. The total angular momentum is $J^2 = l^2 + 3l + 3/2$, where $l = 0, 1, , ,$. The dirac operator has value $l + 3/2$ with the degeneracy $2(l+1)(l+2)$. The degeneracy is divisible by 4 because there are particle-antiparticle and spin degeneracies. Hence, the fermion field can be expanded in the eigenmodes, say $\psi(t,\vec{x}) = \Sigma_{l,m}[b_{lm}(t)Y^{lm}(\vec{x}) + d_{lm}(t)^\dagger Y^{lm}(\vec{x})]$, where m runs over the degenerate states. The linear combination of spin harmonics, $Y^{jm}(\vec{x})$ on S^3, are orthonormal eigenstates of the dirac operator.

Suppose that the total fermionic number in the three sphere is $Q = \Sigma^k_{l=0}(l+1)(l+2)$. The fermionic energy density will be minimum if each shell of harmonics up to the k-th shell is filled only with particles. The expectation value of the energy density and pressure will be O(4)-invariant. The expectation value of the energy density is then

$$\rho_F = \frac{1}{2\pi^2 R^4}\sum_{l=0}^{k}(l + 3/2 + mR)(l+1)(l+2) \tag{3.6}$$

There is also the so-called the vacuum energy density, which is independent of l and proportional to $\hbar/R^4$, which will be negligible in the semiclassical limit.

Now, we are in the position to employ the method in the previous section. The euclidean field equation for a wormhole configuration [5] is

$$\dot{R}^2 + (\rho_F + H^2)R^2 - 1 = 0 \tag{3.7}$$

[5] The naive euclidean field theory of fermions in the path integral formalism is good for the perturbative expansion. However, this theory is not direclty related to tunneling because in this naive theory $\bar{\psi}$ is replaced by $\psi^\dagger$ and $\psi^\dagger$ is not hermitian conjugate of ψ and the spin structures in euclidean and minkowski times are different. To discuss tunneling, we should use either the unknown right euclidean formalism or the energy density expressed in terms of fermion number before going to euclidean time. The later procedure is what we take in the previous section, as well as now.

Let us look at the wormhole size when $Q \approx k^3/3$ is large and so $U(R) = R^2 - (\rho_F + H^2)R^4$ can be approximated by $U(R) = R^2 - k^4/8\pi^2 - mk^3R/6\pi^2 - H^2R^4$.

Let us consider the flat spacetime limit. (Here, we use dimensional quantities.) With $Q = k^3/3$, there are two limits in the parameter space: (1) When $Q \ll 1/(ml_p)^3$, the wormhole size L is $L \approx (3Q)^{2/3}l_p/2\sqrt{2}\pi$. (2) When $Q \gg 1/(ml_p)^3$, $L \approx Qml_p^2/\sqrt{2}\pi > 1/m^2l_p$. The wormhole action can be calculated by subtracting the action of the background spacetime from the fermion counter part of (2.5). If the fermion is massless (massive), the action diverges logarithmically (linearly). The reason for the linear divergence is that the euclidean action for free particles of the total energy E will diverge linearly with the euclidean time T. As shown in [9] for the bosonic case, these divergences are important in representing a wormhole of positive fermion charge $2n$ as an operator ψ^{2n}.

One can ask what is the semiclassical limit of fermions. There is no classical fermion field in the sense that the field strength is much larger than $\sqrt{\hbar}$, since the grassmann variables as the classical fermion fields are anticommuting and take values $0, 1$. However if there is a lot of fermions and fermions are filled upto a certain fermi energy without any holes, one may use the Thomas-Fermi approximation to make sense of the classical limit of other fields in the problem. One example is the fermionic nontopological soliton studied by Friedberg and Lee [13]. In our case also if the fermion number carried by a wormhole is large enough, the wormhole configuration becomes semiclassical (smooth on the planck scale).

In addition, we have seen that there is a fermion contribution to the action, which contradicts the fact that the dirac action is zero when the fermion field equation is satisfied. At least in quantum mechanics of a single grassmann variable, we can see that there is a boundary contribution to the total action [14]

One final question for fermionic wormholes is the spin structure. As wormholes are instantons, the fermion fields are defined first on minkowski spacetime. Euclidean time is introduced to discuss tunneling. One can ask how the spin structure defined on minkowski time is connected to that on euclidean time. For example, there is no majorana fermion in the naive euclidean time even though there could be tunneling involved with this fermion. We do not know any definite answer because the right path integral formalism to deal with tunneling involving fermions is not available. One can at least ponder about the euclidean spin structure, assuming that it is somehow related to our case. In curved spacetime background the spin structure for a dirac fermion can be defined if (1) the spacetime is orientable and (2) the second Stiefel-Whitney class is zero. There are two ways to patch

wormhole necks; orientable and nonorientable. We choose the orientable one. Since a wormhole neck is a three dimensional sphere and the second class is related to the second homotopy group, π_2, the second Stiefel-Whitney class remains zero after patching if it was so before. In multiconnected spacetime, there are many spin structures possible. We do not know which one is related to wormholes, but may assume a unique choice.

4. Conclusion

In this paper, a new class of wormholes is found in a fermion field theory coupled with gravity. This wormhole can be viewed in a flat region as an instanton which interpolates fermions to antifermions. Wormholes violate the fermion number and generate a local effective interaction, $\mathcal{L}_w = \sum \alpha_n \psi^{2n} + h.c.$, where the operator ψ^{2n} of fermion number $2n$ should be invariant under Lorentz transformations.

If the fermion number is locally gauged, wormholes should carry zero charge, losing their *raison d'être*. If there are many species of fermions and the interaction respects some global charges, like baryon number or chirality, wormholes carrying these global charges can exist. Wormholes in turn violate these global charges, inducing proton decay and mass generation for chiral fermions. For proton decay, wormholes would carry any combination of quarks and leptons that has zero gauged charge and one baryon number.

Recently, Coleman and Lee [15] have proposed that wormholes arising in a massive complex scalar field coupled to gravity could be used to solve the cosmological constant problem without generating any giant wormhole catastrophe [16]. Following a similar argument as in [15], one can easily see that wormholes made of fermions can serve the same purpose.

(Note added: After this work had been completed, we became aware that the fermionic wormhole solutions were also recently found by L. Abbott and M. Wise.)

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